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Explicit, Near-Optimal Guidance for Power-Limited Transfers Between Coplanar Circular Orbits

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Within the limits of the classical, two-body, inverse-square-gravity model, explicit guidance solutions are presented for power-limited transfers of a spacecraft between two coplanar circular orbits, performed under the Keplerian class of continuous-thrust programs. The Keplerian class is parameterized by an arbitrary, differentiable, explicit function of time, named the throttling function, and leads to a special type of analytically soluble two-dimensional orbital motion. The primary characteristics of such motion are the resulting qualitative and computational simplicity, exact guidance, conditional power-limited near-optimality, and unlimited two-dimensional maneuverability. It is shown that, for a given transfer duration, the Quadratic subclass of the Keplerian class, for which the throttling function is a quadratic polynomial, can be used to obtain open- and closed-loop solutions that satisfy the boundary conditions exactly, but at the expense of sacrificing some optimality. If, in addition, the transfer duration is sufficiently high, then the Tangential subclass, corresponding to a constant throttling function and tangential thrust, can be used to obtain open- and closed-loop guidance solutions that are identical, very simple to compute in real time, and practically as cheap as the corresponding minimum-fuel solutions.

Nomenclature

B = generalized eccentricity of a transfer, $B \geq 0$
C = generalized orientation of a transfer, $0 \leq C < 2\pi$
E = generalized eccentric anomaly
E_R = dimensional radial thrust-acceleration
E_θ = dimensional transverse thrust-acceleration
e = eccentricity of a conic, $e \geq 0$
g_s = gravitational acceleration at $R = R_s$
H, h = dimensional and nondimensional angular momentum per unit mass, respectively
$J_{\rm PL}$ = power-limited cost [Eq. (45)]
N = number of revolutions about a planet during a transfer
$Q(\tau)$ = throttling function
R, r = dimensional and nondimensional radial distance from a planet
R_s = fixed value of R, selected for the nondimensionalization
t, τ = dimensional and nondimensional time
U, u = dimensional and nondimensional radial velocity component
V, v = dimensional and nondimensional speed of vehicle
V_s = circular orbital speed at $R = R_s$
γ = flight-path angle
ε_r = nondimensional radial thrust-acceleration
ε_θ = nondimensional transverse thrust-acceleration
θ = argument of latitude
μ = gravitational parameter
ω = argument of periapsis, $0 \leq \omega < 2\pi$
$(\,)_f \ldots$ = value at the (final) time $\tau = \tau_f$ (as first subscript)
$(\,)_0 \ldots$ = value at the (initial) time $\tau = 0$ (as first subscript)

I. Introduction

FROM a trajectory-optimization point of view, space propulsion systems usually are divided into two fundamentally different classes.[1–3] For constant specific impulse systems, the power used to eject the propellant is supplied by the propellant itself. For power-limited systems, on the other hand, an independent power source supplies this power. The characteristics of the minimum-fuel trajectories for the two classes are very different. For the former class, one minimizes the characteristic velocity for a maneuver, given by the time integral of the absolute value of the thrust acceleration.[2,4] For the latter class, one minimizes the time integral of the square of the thrust acceleration.[3] The independent power source in power-limited propulsion is usually a solar array or a nuclear reactor, from which the practically achievable thrust levels have so far remained very small.[1] Thus, such propulsion always implies long duration for a maneuver.

The problem of minimum-fuel transfers of a space vehicle between two arbitrary elliptical orbits, under such power-limited propulsion, traditionally has been attacked either numerically,[5,6] by solving the associated two-point boundary value problem, or semi-analytically, by using the method of averaging.[6–8] The numerical method leads to the extremal solution but proves to be computationally intense. The averaging method, on the other hand, especially for two-dimensional motion and sufficiently small initial

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and final eccentricities, successfully predicts the characteristics of the optimal trajectories. Its accuracy, however, decreases as the eccentricities of the elliptical orbits increase, and the method breaks down when either the initial, the final, or both conics are hyperbolic. An extensive literature list for both methods can be found elsewhere.[3,9]

The present paper uses the Keplerian class of thrust programs to propose an alternative, computationally very simple, and nearly optimal method for performing such transfers when the two orbits are coplanar circles. The underlying theory for the Keplerian class was uncovered gradually in three earlier studies.[10–12] It is summarized here in Secs. II and III. These two sections are reproduced from previous work[12] and form a prerequisite for the rest of the paper. The open- and closed-loop problems under consideration are stated explicitly in Sec. IV. In Secs. V and VI, the Quadratic subclass of the Keplerian class is used to obtain analytical solutions to these problems, valid for any transfer duration. In Sec. VII, the Tangential subclass is then used to obtain simplified solutions to the same problems, valid under the assumption of sufficiently high transfer duration. Section VIII compares the power-limited costs corresponding to specific transfer examples, supplied in Secs. V and VII, with the optimal costs corresponding to the same examples, obtained numerically by Haissig.[9] The result turns out to be very favorable for the Tangential subclass. The key points of the work are summarized in Sec. IX.

II. Equations of Motion

Consider the two-dimensional motion of a space vehicle, viewed as a point mass, and moving near an inverse-square gravitational center of attraction (a planet, the sun, etc.). Using polar coordinates R, θ, the differential equations describing such motion can be written as

$$\left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\right) - R\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = -\left(\frac{\mu}{R^2}\right) + E_R \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[R^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\right] = RE_\theta \tag{2}$$

In general, E_R and E_θ arise from any nongravitational forces acting on the vehicle. In the present paper, E_R and E_θ play the role of controls, and are assumed to be attributable solely to thrust. Recalling the definitions for H, U, g_s, and V_s,

$$H = R^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right), \qquad U = \frac{\mathrm{d}R}{\mathrm{d}t} \tag{3}$$

$$g_s = \mu/R_s^2, \qquad V_s = \sqrt{g_s R_s} \tag{4}$$

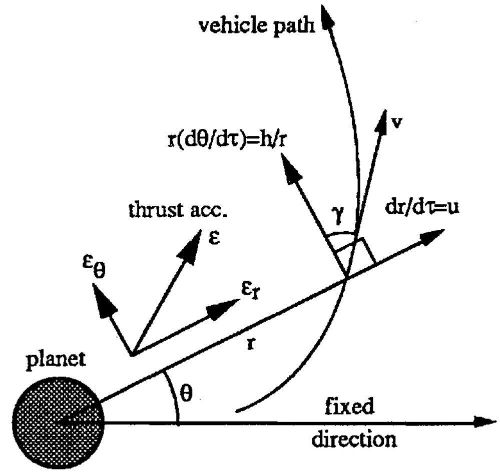

Fig. 1 Polar coordinates and nondimensional state variables.

One can define the nondimensional variables (see Fig. 1):

$$\tau = t(g_s/R_s)^{\frac{1}{2}}, \qquad \varepsilon_r = E_R/g_s, \qquad \varepsilon_\theta = E_\theta/g_s \tag{5}$$

$$r = R/R_s, \qquad u = U/V_s, \qquad v = V/V_s, \qquad h = H/(R_s V_s) \tag{6}$$

Then, using h, u, r, and θ as a set of state variables, Eqs. (1) and (2) assume the equivalent form:

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = r\varepsilon_\theta, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau} = \left(\frac{h^2 - r}{r^3}\right) + \varepsilon_r \tag{7}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = u, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{h}{r^2} \tag{8}$$

III. Keplerian Class of Thrust Programs

For motion under continuous thrust, ε_r and ε_θ in Eqs. (7) are functions of τ. For control-theoretic purposes, however, one can assume that these components depend explicitly on the state variables and τ, namely, $\varepsilon_r = \varepsilon_r(h, u, r, \theta, \tau)$ and $\varepsilon_\theta = \varepsilon_\theta(h, u, r, \theta, \tau)$. Because h, u, r, and θ themselves are functions of τ, this means that ε_r and ε_θ are implicit functions of τ. The interest here is on two-dimensional maneuvers that take place under continuous thrust, starting at $\tau = 0$ and ending at $\tau = \tau_f$. The term thrust program refers to the temporal variation $\varepsilon_r = \varepsilon_r[h(\tau), u(\tau), r(\tau), \theta(\tau), \tau]$, $\varepsilon_\theta = \varepsilon_\theta[h(\tau), u(\tau), r(\tau), \theta(\tau), \tau]$ of the two thrust acceleration components along such a maneuver.

Definition 1: A thrust program is called *Keplerian*, if and only if

$$\varepsilon_\theta = Q/r, \qquad \varepsilon_r = (uQ/h) + (2r\dot{Q}/h) \tag{9}$$

where the variable Q, which from now on is referred to as the *throttling function*, is an explicit, differentiable function of τ [the dot denotes d( )/dτ].

The totality of Keplerian thrust programs defines the Keplerian class of thrust programs, which is parameterized in terms of $Q(\tau)$. When $Q(\tau)$ is identically zero, the thrust is zero and the motion is Keplerian. When $Q(\tau)$ is constant, the thrust vector is tangential to the vehicle's flight path. When $Q(\tau)$ is slowly varying ($\mathrm{d}Q/\mathrm{d}\tau$ is small), the thrust vector is nearly tangential to the vehicle's flight path.

A. Family of Exact Analytical Solutions

It was shown previously[11,12] that all members of the Keplerian class give rise to motion that is analytically soluble. Specifically, if Q is an explicit, differentiable function of τ, and the right-hand sides of Eqs. (7) are forced with the components of the thrust acceleration vector given in Eqs. (9), then the exact solution of Eqs. (7) and (8) is given by the following four quadratures:

$$u = \frac{2rQ + B\sin(\theta - C)}{h}, \qquad r = \frac{h^2}{1 + B\cos(\theta - C)} \tag{10}$$

$$h = h_0 + \int_0^\tau Q\,\mathrm{d}\tau, \qquad \int_0^\tau \frac{\mathrm{d}\tau}{h^3} = \int_{\theta_0}^\theta \frac{\mathrm{d}\theta}{[1 + B\cos(\theta - C)]^2} \tag{11}$$

where h_0, B, C, and θ_0 are four integration constants. This can be seen by differentiating Eqs. (10) and (11) with respect to τ, using Eqs. (9), and verifying that Eqs. (7) and (8) are identically satisfied. The constants h_0 and θ_0 are equal, respectively, to the initial values (at $\tau = 0$) of h and θ. The second of Eqs. (10), describing the trajectory of the vehicle under Keplerian thrust, is identical in form with the trajectory equation corresponding to Keplerian motion.[13] The difference is that h, instead of being constant, is equal to the time integral of Q. If Q is not explicit in τ, then Eqs. (10) still supply two exact quadratures, while obtaining two more quadratures in such a case may or may not be possible, depending on the functional dependence of Q.

The path of the vehicle in $0 \le \tau \le \tau_f$ described by Eqs. (10) and (11) can be referred to as a one-segment transfer trajectory or just as a one-segment transfer, to stress the fact that $Q(\tau)$ is differentiable (and therefore continuous) along it. Clearly, if a piecewise differentiable $Q(\tau)$ is used, then a finite number of such segments can be joined to form a multiple-segment transfer. Hereafter, the term transfer refers solely to one-segment transfers.

From a practical point of view, motion under Keplerian thrust is also characterized by the following two properties[12]: First, for small and slowly varying $Q(\tau)$ [small $\mathrm{d}Q/\mathrm{d}\tau$ compared to $Q(\tau)$], such motion can be expected to be nearly optimal in the power-limited sense (see Appendix A of Ref. 12). Large transfer duration appears as a necessary condition for this to happen. Second, any two-dimensional maneuver can be performed under the Keplerian class. Specifically, as long as h is restricted not to change sign, and the initial and final values of θ do not differ by a multiple of π, any initial state can be driven to any final state in any time interval using a cubic polynomial as $Q(\tau)$ (see Theorem 3 and Appendix C of Ref. 12).

B. Generalized Conics

The constants B and C play qualitatively the same role as the constants e and ω do in Keplerian motion.[13] A transfer thus can be referred to as being circular, elliptic, parabolic, or hyperbolic, depending on whether $B=0$, $0<B<1$, $B=1$, or $B>1$, respectively. In particular, for $B=0$, the exact analytical solution supplied by Eqs. (10) and (11) becomes

$$u = 2hQ, \qquad r = h^2 \tag{12}$$

$$h = h_0 + \int_0^\tau Q\mathrm{d}\tau, \qquad \theta = \theta_0 + \int_0^\tau \frac{\mathrm{d}\tau}{h^3} \tag{13}$$

Also, for zero B, the two thrust acceleration components given in Eqs. (9) become

$$\varepsilon_\theta = \frac{Q}{h^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{1}{h}\right), \qquad \varepsilon_r = 2Q^2 + 2h\dot{Q} = \frac{\mathrm{d}u}{\mathrm{d}\tau} \tag{14}$$

For such transfers, and for a given $Q(\tau)$, it is much more straightforward, compared to the general case of nonzero B, to express h, u, r, and θ as explicit functions of τ. In general, however, to satisfy given boundary conditions, one must use a more complex throttling function to perform a maneuver with a circular transfer than the throttling function that would be required had one chosen to perform it with a noncircular transfer. Moreover, r and h cannot be fixed independently at the endpoints of a circular transfer. Thus, the role played by the constants B and C is a fundamental one and cannot be taken over completely by the throttling function.

C. Polynomial Subclass

Intuitively, for sufficiently large transfer duration, polynomial throttling functions appear as good candidates for satisfying the near-optimality condition mentioned in Sec. III.A.

Definition 2: The subclass of the Keplerian class for which $Q(\tau)$ is an nth-degree polynomial of τ ($n = 0, 1, 2, 3, \ldots$) is called the *Polynomial* class of thrust programs. In particular, for $n = 0, 1, 2$, and 3, the corresponding subclasses of the polynomial class are called the *Tangential*, the *Linear*, the *Quadratic*, and the *Cubic* classes, respectively.

Definition 3: A thrust program is referred to as being tangential if and only if the corresponding thrust acceleration vector is tangent to the vehicle's flight path for $0 \le \tau \le \tau_f$.

Note that $Q(\tau)$ depends on one, two, three, and four constants for the Tangential, the Linear, the Quadratic, and the Cubic classes, respectively. Also, note that a Keplerian thrust program is tangential if and only if it is a member of the Tangential class. This is so because, for a Keplerian thrust program, the ratio $\varepsilon_r/\varepsilon_\theta$ is identically equal to $\tan\gamma$ if and only if $\mathrm{d}Q/\mathrm{d}\tau$ is identically zero.

D. Boundary Conditions

The boundary conditions for a transfer arise from the requirement that the state of the vehicle be continuous at the initial time $\tau = 0$, and the final time $\tau = \tau_f$. If h_0, e_0, and ω_0 are the instantaneous orbital elements at $\tau = 0$, and h_f, e_f, and ω_f are the instantaneous orbital elements at $\tau = \tau_f$, then, comparison of the expressions for u and r, valid for Keplerian motion,[13]

$$u = \frac{e\sin(\theta-\omega)}{h}, \qquad r = \frac{h^2}{1+e\cos(\theta-\omega)} \tag{15}$$

with the corresponding expressions for u and r, supplied in Eqs. (10), implies that the equalities

$$e_0\cos(\theta_0-\omega_0) = B\cos(\theta_0 - C) \tag{16}$$

$$e_f\cos(\theta_f-\omega_f) = B\cos(\theta_f - C) \tag{17}$$

$$e_0\sin(\theta_0-\omega_0) = 2r_0Q_0 + B\sin(\theta_0 - C) \tag{18}$$

$$e_f\sin(\theta_f-\omega_f) = 2r_fQ_f + B\sin(\theta_f - C) \tag{19}$$

are true at the endpoints of a transfer. Also, from Eqs. (11), the continuity of h and θ implies that

$$h_f = h_0 + \int_0^{\tau_f} Q\,\mathrm{d}\tau, \qquad \int_0^{\tau_f}\frac{\mathrm{d}\tau}{h^3} = \int_{\theta_0}^{\theta_f}\frac{\mathrm{d}\theta}{[1+B\cos(\theta-C)]^2} \tag{20}$$

at these endpoints. If the thrust is zero both before $\tau = 0$ and after $\tau = \tau_f$, then, the corresponding motion is Keplerian with orbital elements h_i, e_i, and ω_i ($i = 0, f$). If not, as long as the state of the vehicle is known, one can always determine h_i, e_i, and ω_i ($i = 0$ or f) by means of a local transformation. To summarize, for two-dimensional motion performed under Keplerian thrust, the conditions given by Eqs. (16–20) must always be satisfied at the endpoints of a transfer.

Depending on which of the eight quantities h_i, e_i, ω_i, and θ_i ($i = 0, f$) are left free, there are four broad categories of two-dimensional maneuvers, corresponding to escape problems (h_f, θ_0, and θ_f free), transfer problems (θ_0 and θ_f free), mixed transfers-rendezvous (θ_0 or θ_f free), and rendezvous problems (complete state at $\tau = 0$ and $\tau = \tau_f$ fixed). The final time τ_f itself is always finite, but it may be fixed or left free. $Q(\tau)$ should, in general, depend on a sufficient number of constants, which, together with B, C, and the quantities that are left free, can be used to satisfy Eqs. (16–20).

IV. Statement of the Problem

The problem of transferring under the Keplerian class between two coplanar circular orbits can be stated very simply as follows.

Open-loop problem. Find a throttling function for transferring from a circular orbit at $\tau = 0$ to a coplanar circular orbit at $\tau = \tau_f$. The vehicle's position on the two orbits at $\tau = 0$ and $\tau = \tau_f$ is left free, whereas the transfer time τ_f is fixed.

This problem leads to a throttling function, which, in the absence of state perturbations, guides a vehicle toward the desired final orbit. In the presence of state perturbations, however, and at any intermediate time, the state of the vehicle, projected backward in time using this open-loop throttling function, suggests an initial orbit that is, in general, noncircular. In such a case, to guide a vehicle to its destination, one must use a throttling function that takes it from a specific point of an initial, in general, noncircular orbit, and delivers it to the desired final circular orbit. Repeating this process continuously in $0 < \tau < \tau_f$ leads to a continuous, finite-time, feedback-guidance scheme. If the state of the vehicle can be estimated and the computations for this throttling function can be performed fast enough, then this scheme can be performed in real time, that is, as the actual transfer progresses. The problem of finding this throttling function can be stated as:

Closed-loop problem. Find a throttling function for transferring from a very nearly circular orbit at $\tau = 0$ to a coplanar circular orbit at $\tau = \tau_f$. The vehicle's position on the final circular orbit at $\tau = \tau_f$ is left free, whereas its position on the initial orbit at $\tau = 0$ and the time-to-go τ_f are fixed.

In the real world, circular orbits do not exist. Thus, it is not very realistic to define even the open-loop problem in terms of a circular initial orbit because a vehicle will invariably have to take off from a nearly circular orbit at best. On the other hand, no immediate problem arises with having a desired final orbit that is circular. In transferring to a final orbit that is circular, the position of departure

Table 1 Four open-loop transfer examples

Transfer	1	2	3	4
τ_f	1.9	19	190	1900
D	3.735×10^{-1}	3.735×10^{-3}	3.735×10^{-5}	3.735×10^{-7}
F	-1.966×10^{-1}	-1.966×10^{-4}	-1.966×10^{-7}	-1.966×10^{-10}
θ_f	81.48°	814.88°	8148.81°	81488.1°
N	0.23	2.26	22.64	226.36
J_{PL}	231.7×10^{-3}	12.90×10^{-4}	10.72×10^{-5}	10.69×10^{-6}

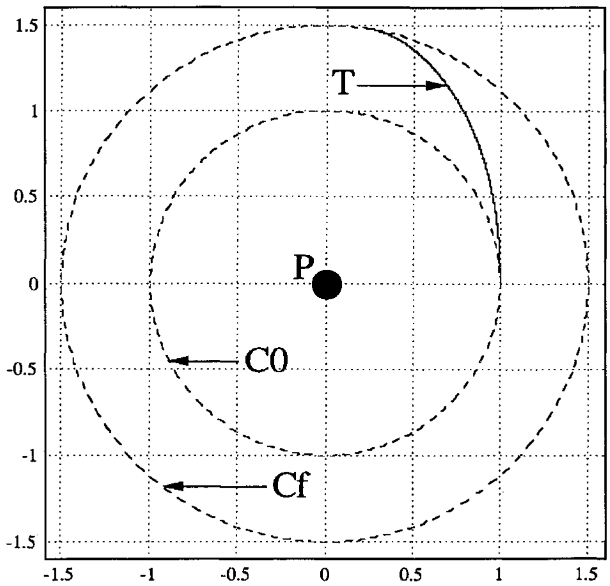

Fig. 2 Plot for transfer 1 in Table 1.

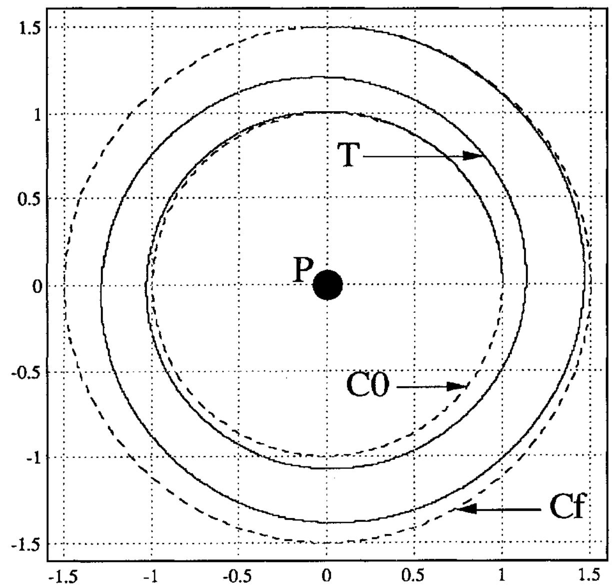

Fig. 3 Plot for transfer 2 in Table 1.

from a very nearly circular initial orbit has practically no effect on the details of the corresponding transfer or power-limited cost. Thus, by considering this position as fixed rather than free, the open- and closed-loop problems can be stated identically. Let the quadruplet $(h_i, e_i, \omega_i, \theta_i)$ denote a conic with orbital elements $h_i > 0$, $e_i \geq 0$, and ω_i, with θ_i denoting the point of departure "from" or arrival "at" the conic. For a circular orbit, the second entry will be zero and the third entry will be a dash. Then, the open- and closed-loop problems can be stated as follows.

Mathematical problem. Find a throttling function $Q(\tau)$ for transferring from a very nearly circular orbit $(h_0, e_0, \omega_0, \theta_0)$ at $\tau = 0$ to a coplanar circular orbit $(h_f, 0, -, \theta_f)$ at $\tau = \tau_f$, where 1) h_0, e_0, ω_0, θ_0, h_f, and τ_f are fixed and 2) θ_f is left free.

V. Exact, Open-Loop Guidance with Circular Transfers ($B = 0$)

For zero e_0 the above mathematical problem affords a very simple solution that uses circular transfers and the Quadratic class. Although it does not correspond to a realistic problem (zero e_0), this solution is important because it supplies quite an accurate first guess for the value of θ_f that can be used during the computation of the closed-loop solution (Sec. VI). When e_0 and e_f are both zero, a very simple way of satisfying Eqs. (16–19) consists of selecting

$$B = 0, \qquad Q_0 = Q_f = 0 \tag{21}$$

Because of these conditions on Q, a constant or linear throttling function will have to vanish identically. Thus, using instead a quadratic throttling function,

$$Q = F\tau^2 + D\tau + A \tag{22}$$

where F, D, and A are constants, and the first of Eqs. (11) or (13), one finds

$$h = (F\tau^3/3) + (D\tau^2/2) + A\tau + h_0 \tag{23}$$

The two conditions on Q, and the condition $h = h_f$ when $\tau = \tau_f$ [first of Eqs. (20)], imply that

$$A = 0, \qquad D = \frac{6(h_f - h_0)}{\tau_f^2}, \qquad F = -\frac{6(h_f - h_0)}{\tau_f^3} \tag{24}$$

The open-loop problem now has been solved completely. The open-loop throttling function is

$$Q = 6\left(\frac{h_f - h_0}{\tau_f}\right)\left(\frac{\tau}{\tau_f}\right)\left(1 - \frac{\tau}{\tau_f}\right) \tag{25}$$

This transfers from a precisely circular initial, to a coplanar, precisely circular final orbit in a time interval τ_f. The corresponding transfer is itself a circular one ($B = 0$), and h along it varies as

$$h = -(h_f - h_0)\left[2(\tau/\tau_f)^3 - 3(\tau/\tau_f)^2\right] + h_0 \tag{26}$$

Clearly, h goes through its two extrema at $\tau = 0$ and $\tau = \tau_f$ and remains positive in $0 \leq \tau \leq \tau_f$. With $Q(\tau)$, $h(\tau)$ known, Eqs. (14) specify the open-loop thrust acceleration program. The components r and u are given by Eq. (12). The polar coordinate θ is given by the second of Eqs. (13). Because of the circular symmetry, the transfer can start from any point on the initial orbit, and so, there is no loss of generality in taking θ_0 as zero. Then, with $\tau = \tau_f \zeta$, $d\tau = \tau_f\, d\zeta$, from the second of Eqs. (20), θ_f is

$$\theta_f = \int_0^1 \frac{\tau_f\, d\zeta}{\left[-2(h_f - h_0)\zeta^3 + 3(h_f - h_0)\zeta^2 + h_0\right]^3} \tag{27}$$

Note that θ_f is proportional to the transfer duration τ_f and that $N = \theta_f / 2\pi$ is the number of revolutions about the planet during the transfer. Table 1 supplies the details for four such transfer examples. Each example represents a transfer from an initial orbit with $h_0 = 1$, $e_0 = 0$, to a final orbit with $h_f = (1.5)^{1/2} = 1.2247\ldots$, $e_f = 0$. The examples correspond to four values of τ_f. Constants A and B, as well as θ_0, are zero for all four examples. The last row of

Table 1 supplies the exact power-limited cost J_{PL}. Note how J_{PL} decreases with increasing τ_f. The transfers in Table 1 corresponding to $\tau_f = 1.9$ and $\tau_f = 19$ are shown in Figs. 2 and 3, respectively.

VI. Exact, Closed-Loop Guidance with Nearly Circular Transfers ($B \approx 0$)

Recall the mathematical problem defined in Sec. IV and the boundary conditions, Eqs. (16–20). With zero e_f, but nonzero e_0, one can solve Eqs. (16) and (17) for $B\sin C$, $B\cos C$:

$$B\sin C = -\frac{e_0\cos\theta_f\cos(\theta_0-\omega_0)}{\sin(\theta_f-\theta_0)} \tag{28}$$

$$B\cos C = \frac{e_0\sin\theta_f\cos(\theta_0-\omega_0)}{\sin(\theta_f-\theta_0)} \tag{29}$$

Under the conventions $B \geq 0$, $0 \leq C < 2\pi$, and for $\theta_f - \theta_0$ not equal to a multiple of π, Eqs. (28) and (29) uniquely determine B and C. In particular, C differs from θ_f by either $\pi/2$ or $3\pi/2$. Using $\varphi = \theta - \omega_0$, Eqs. (18) and (19), with zero e_f, then imply that

$$Q_0 = -\frac{e_0\cos\varphi_f(1+e_0\cos\varphi_0)}{2h_0^2\sin(\varphi_f-\varphi_0)} \tag{30}$$

$$Q_f = -\frac{e_0\cos\varphi_0}{2h_f^2\sin(\varphi_f-\varphi_0)} \tag{31}$$

Because θ_f is left free to satisfy the above two conditions on Q and the two conditions in Eqs. (20), one can again select a quadratic Q, supplying just the right number of constants. Then, Q and h are still given by Eqs. (22) and (23), but now, instead of Eqs. (24), constants A, D, and F are found as

$$A = Q_0, \qquad D = -\frac{2(Q_f+2Q_0)}{\tau_f} + \frac{6(h_f-h_0)}{\tau_f^2} \tag{32}$$

$$F = \frac{3(Q_f+Q_0)}{\tau_f^2} - \frac{6(h_f-h_0)}{\tau_f^3} \tag{33}$$

Through Eqs. (28–33), the constants B, C, A, D, and F become functions of θ_f. The value(s) of θ_f corresponding to a solution are the ones for which the second of Eqs. (20) is satisfied.

A. Generalized Kepler's Condition

The second of Eqs. (11) plays qualitatively the same role that the classical Kepler's equation plays in Keplerian motion.[13] When $B = 0$ or $B = 1$, integration of the right-hand side of this equation is simple.[14] When $0 < B < 1$ or $B > 1$, in analogy with Keplerian motion,[13] such an integration can be performed by defining, respectively, generalized eccentric or hyperbolic anomalies. With $\xi = \theta - C$,

$$\int_0^\tau \frac{d\tau}{h^3} = \int_{\xi_0}^{\xi} \frac{d\xi}{(1+B\cos\xi)^2} \tag{34}$$

For $0 < B < 1$, the change of variable from ξ to the generalized eccentric anomaly E reads[12]

$$\cos E = \frac{\cos\xi + B}{1+B\cos\xi}, \qquad \sin E = \frac{\sqrt{1-B^2}\sin\xi}{1+B\cos\xi} \tag{35}$$

Then, Eq. (34) can be reduced to

$$\int_0^\tau \frac{d\tau}{h^3} = \frac{1}{(1-B^2)^{\frac{3}{2}}}[E - E_0 - B(\sin E - \sin E_0)] \tag{36}$$

Let now $0 \leq E_0 < 2\pi$, and use the definitions

$$I_f = \int_0^{\tau_f}\frac{d\tau}{h^3}, \qquad b = (1-B^2)^{\frac{3}{2}}, \qquad E_f = E_{fP} + 2k\pi \tag{37}$$

where $0 \leq E_{fP} < 2\pi$ and k is any nonnegative integer. Then, Eq. (36) evaluated at $\tau = \tau_f$ becomes

$$(1/2\pi)[I_f b + B(\sin E_{fP} - \sin E_0) - (E_{fP} - E_0)] = k \tag{38}$$

Equation (38) is the same as the second of Eqs. (20), and from now on it will be referred to as the Generalized Kepler's condition (GKC). Through Eqs. (28–33) and (35), the left-hand side of Eq. (38) is an implicit function of E_{fP}. To solve the closed-loop problem, one must first find the values of E_{fP} in $0 \leq E_{fP} < 2\pi$ for which the left-hand side of Eq. (38) is equal to a nonnegative integer.

B. Solution Methodology

In general, the above closed-loop problem has multiple solutions. It will be seen by an example that one among those can be viewed as the solution of the open-loop problem, perturbed by the initial small, nonzero e_0. Not surprisingly, this particular solution is also the cheapest one among all solutions in the power-limited sense. Thus, it is the solution that one seeks when solving the closed-loop problem. To find it, one proceeds roughly as follows:

1) The quantities h_0, e_0, ω_0, θ_0, h_f, and τ_f are given.

2) One evaluates the right-hand side of Eq. (27) and adds to it the value of θ_0 [Eq. (27) is valid for zero θ_0]. The result is a first-guess θ_{f1} for θ_f, valid for zero e_0.

3) θ_{f1} is written as $\theta_{f1} = \theta_{f1P} + 2k_1\pi$, with $0 \leq \theta_{f1P} < 2\pi$, and k_1 a nonnegative integer.

4) For small e_0 the solution θ_{fS} lies close to θ_{f1}. Thus, one guesses a value θ_{f2} close to θ_{f1}.

5) One computes all of the relevant quantities (B, C, A, D, F, etc.) corresponding to $\theta_f = \theta_{f2}$.

6) One evaluates the left-hand side of GKC, Eq. (38). If the result equals k_1, or $k_1 + 1$, or $k_1 - 1$, then θ_{f2} is the solution θ_{fS}. If not, one guesses a new value θ_{f3} and repeats steps (5) and (6).

By guessing alternately values for θ_f that lie above or below θ_{f1}, one can trap the solution θ_{fS} between two such guesses. A numerical scheme then can be used to find θ_{fS}. The only complication that such a scheme will have to accommodate is that the value of k corresponding to the solution may jump by $+1$ or -1 with respect to the value k_1 corresponding to θ_{f1}. Computing time can be saved if one is willing to develop an analytical formula for I_f.

The totality of solutions can be seen by plotting the left-hand side of GKC, Eq. (38), as a function of the principal value θ_{fP} of θ_f, for $0 < \theta_{fP} < 2\pi$. Every point at which such a plot intersects a nonnegative integer represents a candidate solution. This plot is presented in Fig. 4 for example 2 of Table 1. The plot corresponds to $h_0 = 1.01$, $e_0 = 0.01$, $\theta_0 = 0$, $\omega_0 = 0$, with h_f and e_f the same as in Table 1. The dots on the plot indicate all possible solutions. The characteristics for four among these are given in Table 2. Transfer 5 in Table 2 is just the perturbed solution corresponding to transfer 2 in Table 1. It is also the cheapest one among all possible solutions in Fig. 4.

Solutions in Fig. 4 abound where $\theta_{fP} - \theta_0$ is close to $0, \pi$, or 2π. There is, however, a finite number of them. The reason is that,

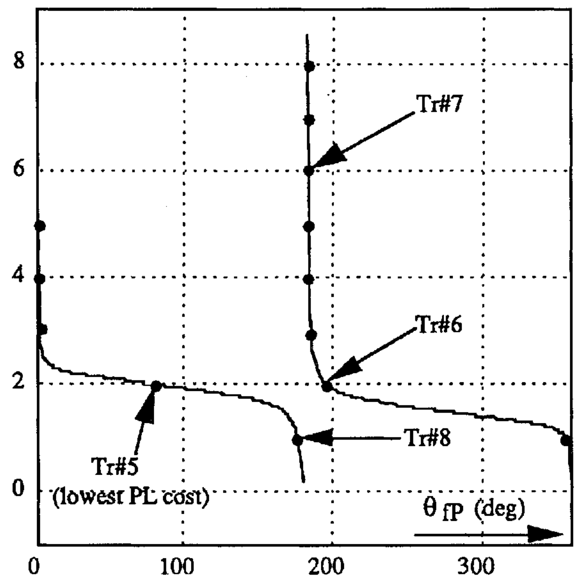

Fig. 4 Left-hand side of Eq. (38) vs θ_{fP}.

Table 2 Four closed-loop transfer examples

Transfer	5	6	7	8
τ_f	19	19	19	19
A	-1.192×10^{-3}	-1.870×10^{-2}	-1.179×10^{-1}	6.841×10^{-2}
D	4.181×10^{-3}	6.135×10^{-3}	2.003×10^{-2}	-5.971×10^{-3}
F	-2.263×10^{-4}	-2.350×10^{-4}	-5.074×10^{-4}	-3.143×10^{-6}
B	1.029×10^{-2}	3.908×10^{-2}	2.384×10^{-1}	1.385×10^{-1}
C	346.49°	284.88°	272.47°	85.85°
θ_f	796.49°	914.88°	2342.47°	535.85°
N	2.21	2.54	6.51	1.49
J_{PL}	13.27×10^{-4}	20.71×10^{-4}	618.37×10^{-4}	88.18×10^{-4}

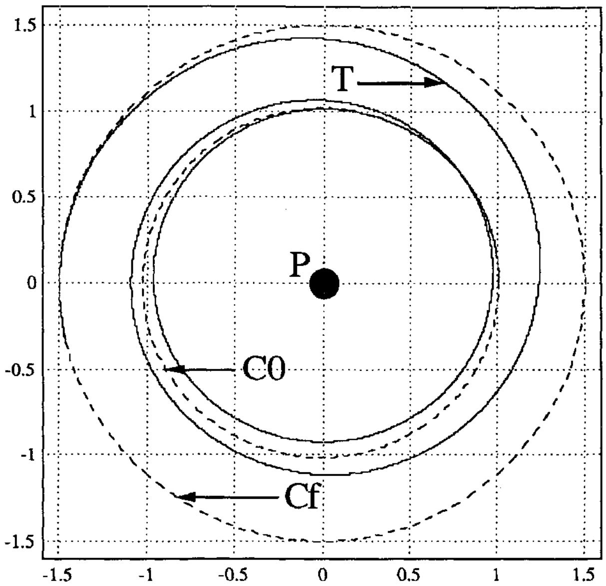

Fig. 5 Plot for transfer 6 in Table 2 and Fig. 4.

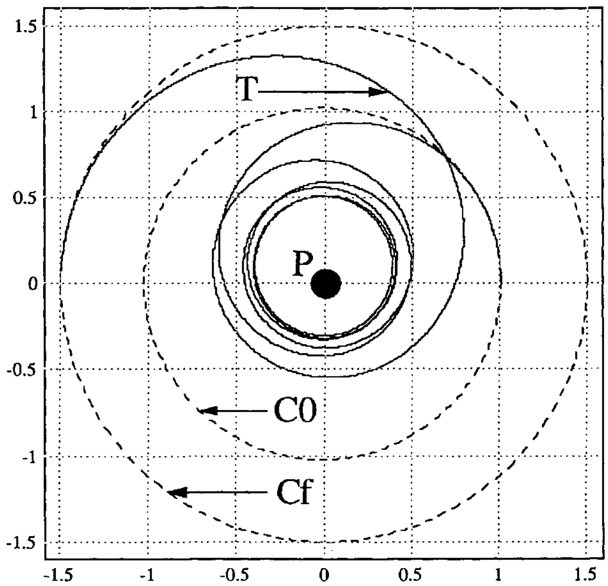

Fig. 6 Plot for transfer 7 in Table 2 and Fig. 4.

as $\theta_{fP} - \theta_0$ approaches $0, \pi$, or 2π, the generalized eccentricity B eventually exceeds unity. Full revolutions about the planet are then precluded during a transfer. As e_0 tends to zero, the number of solutions tends to infinity. For zero e_0, only one solution remains, which corresponds to the open-loop problem of Sec. V. For solutions with large values of k, the vehicle tends to stay mostly close to, and revolve many times about, the planet during the transfer. Opposite trends hold for solutions with small values of k. Examples 6, 7, and 8 in Table 2, and the corresponding transfers, plotted in Figs. 5, 6,

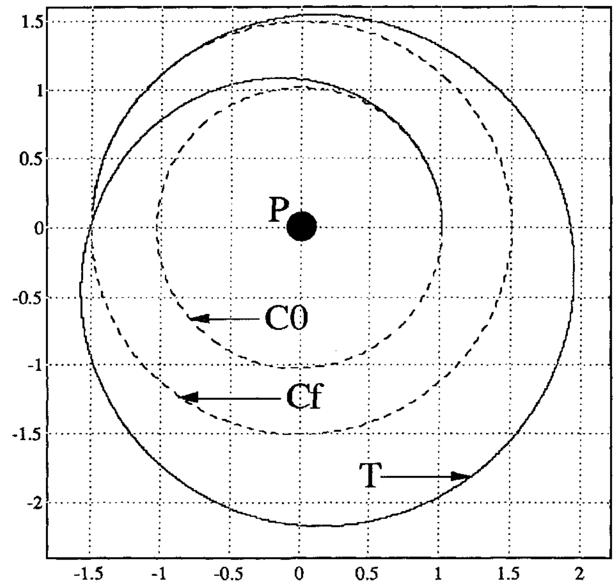

Fig. 7 Plot for transfer 8 in Table 2 and Fig. 4.

and 7, clearly depict these trends. Table 2 shows that, compared to example 5, examples 6–8 correspond to very expensive transfers.

VII. Nearly Optimal Guidance Under the Tangential Class

Demanding that e_f be exactly zero is unrealistic because precisely circular orbits do not exist in the real world. It is thus much more meaningful to relax this condition and try to find transfers to practically circular orbits with very small e_f. The mathematical problem thus can be modified as follows: Find a throttling function $Q(\tau)$ for transferring from a very nearly circular orbit $(h_0, e_0, \omega_0, \theta_0)$ at $\tau = 0$ to a coplanar, very nearly circular orbit $(h_f, e_f, \omega_f, \theta_f)$ at $\tau = \tau_f$, where 1) $h_0, e_0, \omega_0, \theta_0, h_f$, and τ_f are fixed and 2) e_f, ω_f, and θ_f are left free.

For sufficiently large transfer duration, this problem accepts the remarkably simple solution

$$Q(\tau) = A = (h_f - h_0)/\tau_f = \text{constant} \tag{39}$$

To show this, consider a vehicle that takes off at $\tau = 0$ from any point of the initial very nearly circular orbit. The thrust program used is defined by Eqs. (9) and (39). The thrust is kept on until $\tau = \tau_f$ and then turned off. From the first of Eqs. (11), after $\tau = \tau_f$, the vehicle is on a final conic with angular momentum h_f. In addition, for practically zero e_0, and for sufficiently large τ_f, the eccentricity e_f is practically zero, and the final conic is essentially the desired circular orbit. To see this, start with Eqs. (16–19), valid at the endpoints of such a transfer. Rearrange Eq. (18) and then square both sides of Eqs. (16) and (18) and add to get

$$B^2 = e_0^2 + 4A^2 r_0^2 - 4Ae_0 r_0 \sin(\theta_0 - \omega_0) \tag{40}$$

Table 3 Four examples of takeoff from a circular orbit, under the Tangential class

Transfer	9	10	11	12
τ_f	1.9	19	190	1900
e_f	3.99×10^{-1}	4.18×10^{-2}	5.69×10^{-3}	4.43×10^{-4}
ω_f	44.59°	33.91°	70.71°	37.04°
A	1.183×10^{-1}	1.183×10^{-2}	1.183×10^{-3}	1.183×10^{-4}
B	2.366×10^{-1}	2.366×10^{-2}	2.366×10^{-3}	2.366×10^{-4}
C	90°	90°	90°	90°
θ_f	98.74°	809.41°	8067.97°	80734.88°
N	0.27	2.25	22.41	224.26
J_{PL}	12.00×10^{-3}	9.02×10^{-4}	8.98×10^{-5}	8.98×10^{-6}

The initial radial distance r_0 is finite, given by

$$r_0 = \frac{h_0^2}{1 + e_0 \cos(\theta_0 - \omega_0)} \tag{41}$$

From Eq. (40), for sufficiently large τ_f, and thus sufficiently small A, B is sufficiently small and of the same order as e_0. Next, square both sides of Eqs. (17) and (19) and add to get

$$e_f^2 = B^2 + 4A^2 r_f^2 + 4ABr_f \sin(\theta_f - C) \tag{42}$$

The final radial distance r_f is

$$r_f = \frac{h_f^2}{1 + B\cos(\theta_f - C)} \tag{43}$$

Then, for $0 < B < 1$, which is the case here, r_f is finite. Again, from Eqs. (40–43), for sufficiently large τ_f, and thus sufficiently small A, B is sufficiently small and of the same order as e_0, and e_f is sufficiently small and of the same order as B. This concludes the proof.

Note that at any time τ along such a transfer, with the subscript f deleted, relationships identical to Eqs. (16–20), (42), and (43) are valid. Thus, for sufficiently small A, the eccentricity of the instantaneous Keplerian conic at each point of the transfer is sufficiently small. This implies that the instantaneous conic is very nearly circular. Accordingly, the open- and closed-loop strategies associated with the practical problem are identical, and they are both based on the expression for Q given in Eq. (39). For the open-loop problem, τ_f plays the role of the final time, or transfer duration. For the closed-loop problem, τ_f plays the role of the time-to-go. From Eq. (9), the thrust acceleration components corresponding to Eq. (39) are

$$\varepsilon_\theta = A/r, \qquad \varepsilon_r = (uA)/h \tag{44}$$

The constants B and C on such a transfer are fixed by Eqs. (16) and (18). The state of the vehicle along the transfer is fixed by Eqs. (10) and (11). The open-loop thrust acceleration components are then generated by Eqs. (44). On the other hand, if at any instant along the transfer, an estimate of the state is available, then the closed-loop components are formed from Eqs. (44) using that state. It will be seen in Sec. VIII that, for sufficiently large τ_f, the guidance suggested by Eq. (39) essentially represents the optimal power-limited guidance between two coplanar circular orbits.

Table 3 supplies four such transfer examples, for various values of τ_f. In each example, $h_f = (1.5)^{1/2}$, $h_0 = 1$, and $e_0 = 0$. The vehicle is assumed to take off from the point $\theta_0 = 0$ under a constant Q supplied by Eq. (39). The thrust is turned off at $\tau = \tau_f$. After that, the vehicle coasts along a conic with $h_f = (1.5)^{1/2} = 1.2247$. It is seen from the values of e_f that, as τ_f increases, for the fixed h_f, the final orbit tends to become more and more circular. For example 9, e_f is about 0.4, and the corresponding orbit is noncircular. For examples 10, 11, and 12, on the other hand, e_f is so small that the corresponding orbits can be considered practically circular. Thus, these three examples are directly comparable with examples 2, 3, and 4, respectively, of Table 1. Note from the last row of Table 3 that the power-limited cost decreases as the transfer duration increases. Comparing with the last row of Table 1, it is seen that for τ_f equal to 19, 190, and 1900, practically the same transfers can be performed under the Tangential class at lower cost than their counterparts under the Quadratic class. The transfer in Table 3 for $\tau_f = 19$ is given in Fig. 8 (compare with Fig. 3). Note that the results of this section have already been extended to a more general class of transfers.[15]

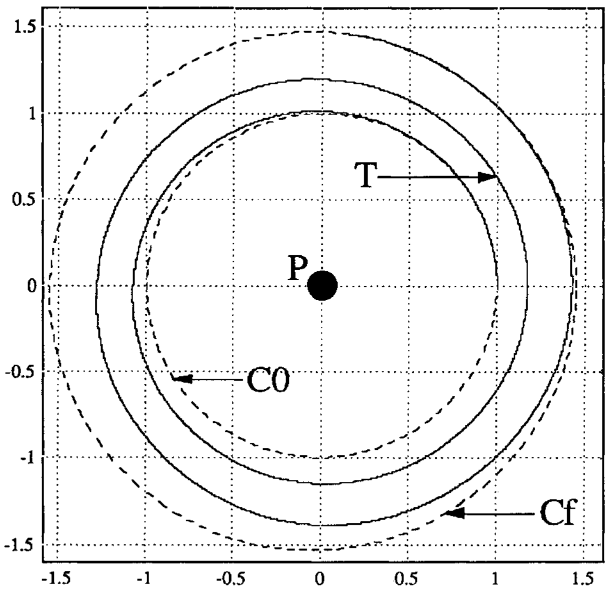

Fig. 8 Plot for transfer 10 in Table 3.

VIII. Comparisons for the Power-Limited Cost

The power-limited cost for any two-dimensional maneuver is given by[3]

$$J_{\text{PL}} = \frac{1}{2}\int_0^{\tau_f} \left(\varepsilon_\theta^2 + \varepsilon_r^2\right) \mathrm{d}\tau \tag{45}$$

The cost supplied in the last row of Tables 1, 2, and 3 is the exact power-limited cost for the corresponding transfers. It was computed by numerically evaluating the integral on the right-hand side of Eq. (45) using the components ε_r, ε_θ given, respectively, by Eqs. (14), (9), and (44). In addition, for constant Q [Eq. (39)], one can derive an approximate expression for J_{PL}, based on the fact that for large τ_f the approximations $B \approx 0$ and $r \approx h^2$ are valid [see Eqs. (40) and (12)]. This is shown in the Appendix [see Eq. (A9)], and it leads to the following very simple expression for J_{PL}, valid for transfers between two nearly circular orbits, performed under the Tangential class:

$$J_{\text{PL}} = \left(\frac{h_f - h_0}{6\tau_f}\right)\left(\frac{1}{h_0^3} - \frac{1}{h_f^3}\right) \tag{46}$$

The costs presented in Tables 1 and 3, as well as the validity of Eq. (46), will now be checked against the corresponding optimal costs obtained by Haissig.[9] Haissig used the same nondimensional cost as in Eq. (45). By considering practically the same transfer problems corresponding to $\tau_f = 19$, 190, and 1900 in Tables 1 and 3, and by viewing these as power-limited optimization problems, she obtained

Table 4 Power-limited cost comparisons

Transfer	2, 10	3, 11	4, 12
τ_f	19	190	1900
J_{PL} (averaged)	9.66×10^{-4}	9.5×10^{-5}	9.5×10^{-6}
J_{PL} (extremal)	9.51×10^{-4}	9.5×10^{-5}	9.5×10^{-6}
e_f (desired)	5.00×10^{-2}	5.5×10^{-3}	6×10^{-4}
e_f (achieved)	6.71×10^{-2}	10.08×10^{-3}	11.17×10^{-4}
Δe_f (error)	1.71×10^{-2}	4.58×10^{-3}	5.17×10^{-4}
J_{PL} (Tangential)	9.02×10^{-4}	8.98×10^{-5}	8.98×10^{-6}
J_{PL} [Eq. (46)]	8.98×10^{-4}	8.98×10^{-5}	8.98×10^{-6}
e_f (achieved)	4.18×10^{-2}	5.69×10^{-3}	4.43×10^{-4}
J_{PL} (Quadratic)	12.90×10^{-4}	10.72×10^{-5}	10.69×10^{-6}

two solutions in each of the three cases. The first one was the approximate solution based on the analytical method of averaging. The second was the extremal solution. It was obtained by solving numerically the associated two-point boundary-value problem. Haissig's results (see Tables 5.6 and 5.7 of Ref. 9) are reproduced in rows 3–7 of Table 4. J_{PL} (averaged) and J_{PL} (extremal) are, respectively, the costs obtained by the two solutions. The target eccentricity is e_f (desired), which, to avoid numerical singularities, was taken as small but nonzero (see Table 5.6 of Ref. 9). The extremal solution [see Table 5.7 and Eq. (5.4) of Ref. 9] satisfied this condition very well [e_f (achieved)], resulting in only a small error [Δe_f (error)].

The results from the present paper from Table 3 and Eq. (46) are presented as J_{PL} (Tangential), J_{PL} [Eq. (46)], and e_f (achieved) in rows 8, 9, and 10, respectively, of Table 4. Strictly speaking, because of the differences in e_f (achieved), these results are not directly comparable with the results of Haissig.[9] Practically, however, in both cases, the transfers are between the same two circular orbits. In this respect, Table 4 implies that the thrust program given by Eq. (44) is practically as good as the optimal thrust program between the two circular orbits. In fact, it is seen that the transfers supplied in Table 3 for $\tau_f = 19$, 190, and 1900 have lower cost than those found by Haissig.[9] The reason for this is presumably that Haissig terminated her iterations before exactly satisfying the final condition on e_f. Moreover, Table 4 also suggests that Eq. (46) supplies a very accurate prediction for the cost as the transfer duration τ_f becomes sufficiently large. The last row of Table 4 reproduces J_{PL} from Table 1. It shows that the Quadratic class is nearly optimal or suboptimal at best, to be used only when very accurate satisfaction of the condition of zero e_f is desired.

The conditions for the power-limited near-optimality of the Keplerian class were discussed in Appendix A of Ref. 12. The near-optimality of the Tangential class can be deduced from those results and was shown explicitly in Ref. 11. Specifically, for large τ_f and small (and constant) Q, the Tangential class satisfies the first-order necessary conditions for power-limited optimality extremely well (see Ref. 11, Sec. 4.6.5, p. 75). Combining these observations with the comparisons supplied in Table 4 reinforces the implication that for sufficiently large transfer duration, or, sufficiently small thrust acceleration levels, the type of guidance presented in Sec. VII is practically the optimal power-limited guidance between two circular or very nearly circular orbits, and the cost supplied by Eq. (46) is the corresponding optimal power-limited cost.

IX. Conclusions

The Keplerian class of thrust programs implies at least two computationally very simple methods of transferring a spacecraft, under power-limited propulsion, between two coplanar, circular, or very nearly circular orbits. The thrust program for the first method is defined by a quadratic throttling function. It can be used for very accurate satisfaction of the zero eccentricity conditions, but at the expense of sacrificing some optimality. The corresponding cost decreases with increasing transfer duration, implying that a sufficiently large transfer duration may result in low, and possibly acceptable, levels of fuel consumption. The thrust program for the second method is defined by a constant throttling function, equal to the difference between the angular momenta of the two orbits, divided by the transfer duration. Such guidance can be used to transfer between two very nearly circular orbits, having very small, but apart from that, free eccentricities. Explicit cost comparisons carried out in this paper, combined with a previous examination of the necessary conditions for optimality, suggest that, for sufficiently large transfer duration, or sufficiently small thrust acceleration levels, this second type of guidance results in practically the same cost as the cost corresponding to the minimum-fuel, power-limited guidance between the two orbits.

Appendix: Derivation of Eq. (46)

The nondimensional mechanical energy S (kinetic plus potential) per unit mass of a vehicle is

$$S = (v^2/2) - (1/r), \qquad v^2 = u^2 + (h^2/r^2) \tag{A1}$$

For a transfer under the Tangential class, from Eqs. (44), the corresponding thrust acceleration is

$$\varepsilon = \left(\varepsilon_\theta^2 + \varepsilon_r^2\right)^{\frac{1}{2}} = [(A^2/r^2) + (A^2u^2/h^2)]^{\frac{1}{2}} = Av/h \tag{A2}$$

Differentiating Eqs. (A1) with respect to τ, and using Eqs. (7–9) and (A2), one obtains

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \frac{Av^2}{h} = \varepsilon v, \qquad \frac{\mathrm{d}S}{\mathrm{d}h} = \frac{v^2}{h} \tag{A3}$$

From Eq. (45), the power-limited cost associated with such a transfer is

$$J_{PL} = \frac{1}{2}\int_0^{\tau_f} \varepsilon^2\,\mathrm{d}\tau = \frac{A}{2}\int_{h_0}^{h_f} \frac{v^2\,\mathrm{d}h}{h^2} = \frac{A}{2}\int_{S_0}^{S_f} \frac{\mathrm{d}S}{h} \tag{A4}$$

Integrating by parts in Eq. (A4) one obtains

$$J_{PL} = \frac{A}{2}\left(\frac{S_f}{h_f} - \frac{S_0}{h_0}\right) + \frac{A}{2}\int_{h_0}^{h_f} \frac{S\,\mathrm{d}h}{h^2} \tag{A5}$$

With the definition of S, and the middle expression for J_{PL} in Eq. (A4), one can write Eq. (A5) as

$$J_{PL} = A\left(\frac{S_f}{h_f} - \frac{S_0}{h_0}\right) - A\int_{h_0}^{h_f} \frac{\mathrm{d}h}{rh^2} \tag{A6}$$

At each point of the transfer, S is related to the instantaneous orbital elements through[13]

$$S = (e^2 - 1)/2h^2 \tag{A7}$$

Then, Eq. (A6) can be written as

$$J_{PL} = \frac{A}{2}\left(\frac{e_f^2 - 1}{h_f^3} - \frac{e_0^2 - 1}{h_0^3}\right) - A\int_{h_0}^{h_f} \frac{\mathrm{d}h}{rh^2} \tag{A8}$$

On a circular transfer, $B = 0$ and $r = h^2$ [see second of Eqs. (12)], and from Eq. (A8),

$$J_{PL} = \frac{A}{6}\left(\frac{3e_f^2 - 1}{h_f^3} - \frac{3e_0^2 - 1}{h_0^3}\right) \tag{A9}$$

For motion under the Tangential class between two very nearly circular orbits, as τ_f tends to infinity, B tends to zero and Eq. (A9) with $A = (h_f - h_0)/\tau_f$ is asymptotically valid. If, in addition, one neglects the squares of the initial and final eccentricities, one obtains Eq. (46), which, as τ_f tends to infinity, supplies an asymptotic representation of the power-limited cost.

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